International Journal of Geometric Methods in Modern Physics
Vol. 14, No. 9 (2017) 1750119 (14 pages)

DOI: 10.1142/S0219887817501195

Pseudo B-symmetric manifolds

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Received 21 January 2017
Accepted 27 March 2017
Published 2 May 2017

Dedicated to the memory of W. Roter

In this paper, we introduce a new tensor named B-tensor which generalizes the Z-tensor introduced by Mantica and Suh [Pseudo Z symmetric Riemannian manifolds with harmonic curvature tensors, *Int. J. Geom. Methods Mod. Phys.* **9**(1) (2012) 1250004]. Then, we study pseudo-B-symmetric manifolds $(PBS)_n$ which generalize some known structures on pseudo-Riemannian manifolds. We provide several interesting results which generalize the results of Mantica and Suh [Pseudo Z symmetric Riemannian manifolds with harmonic curvature tensors, *Int. J. Geom. Methods Mod. Phys.* **9**(1) (2012) 1250004]. At first, we prove the existence of a $(PBS)_n$. Next, we prove that a pseudo-Riemannian manifold is B-semisymmetric if and only if it is Ricci-semisymmetric. After this, we obtain a sufficient condition for a $(PBS)_n$ to be pseudo-Ricci symmetric in the sense

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of Deszcz. Also, we obtain the explicit form of the Ricci tensor in a $(PBS)_n$ if the B-tensor is of Codazzi type. Finally, we consider conformally flat pseudo-B-symmetric manifolds and prove that a $(PBS)_n (n > 3)$ spacetime is a *pp*-wave under certain conditions.

Keywords: Pseudo-symmetric manifolds; pseudo-Ricci symmetric manifolds; pseudo-Z-symmetric manifolds; pseudo-B-symmetric manifolds.

Mathematics Subject Classification 2010: 53C25

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of pseudo-Riemannian symmetric spaces was initiated in the late 20s by Cartan [6], who, in particular, obtained a classification of those spaces. Let $(M^n, g), (n = \dim M)$ be a pseudo-Riemannian manifold, i.e. a manifold M with the pseudo-Riemannian metric g, and let ∇ be the Levi-Civita connection of (M^n, g). A pseudo-Riemannian manifold is called locally symmetric [6] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g).

As a generalization of Ricci symmetric manifolds ($\nabla_k R_{ij} = 0, R_{ij}$ is the Ricci tensor), Chaki [3] introduced pseudo-Ricci symmetric manifolds. A non-flat pseudo-Riemannian manifold $(M^n, g), (n > 2)$ is said to be a pseudo-Ricci symmetric manifold if its curvature tensor satisfies the condition

$$\nabla_k R_{ij} = 2A_k R_{ij} + A_i R_{kj} + A_j R_{ik}, \tag{1.1}$$

where A_i is a nonzero 1-form. ∇_k denotes the covariant differentiation with respect to the metric tensor g. The 1-form A_i is called the associated 1-form of the manifold. If $A_i = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan. An n-dimensional pseudo-Ricci symmetric manifold is denoted by $(PRS)_n$.

In 1993, Tamassy and Binh [26] introduced weakly Ricci symmetric manifolds. It may be mentioned that a pseudo-Ricci symmetric manifold is a particular case of a weakly Ricci symmetric manifold. In a recent paper [19], Mantica and Suh introduced pseudo-Z-symmetric manifolds which is denoted by $(PZS)_n$. It is a generalization of the notion of pseudo-Ricci symmetric manifolds, pseudo-projective-Ricci symmetric manifolds [5]. A $(0, 2)$ symmetric tensor is a generalized Z-tensor if

$$Z_{ij} = R_{ij} + \phi g_{ij}, \tag{1.2}$$

where ϕ is an arbitrary scalar function. The scalar Z is obtained by transvecting (1.2) with g^{ij} as follows:

$$Z = R + n\phi. \tag{1.3}$$

In this paper, we introduce a $(0, 2)$ symmetric tensor B_{ij} as follows:

$$B_{ij} = aR_{ij} + bRg_{ij}, \tag{1.4}$$

where a and b are nonzero arbitrary scalar functions and R is the scalar curvature. The scalar B is obtained by transvecting (1.4) with g^{ij} as follows:

$$B = (a + nb)R. \tag{1.5}$$

Pseudo-Z-symmetric, weakly Z-symmetric and recurrent Z forms on pseudo-Riemannian manifolds have been studied in ([19–21]), respectively.

Inspired by these works, we introduce a new type of manifold called pseudo-B-symmetric manifolds. A manifold is called pseudo-B-symmetric and denoted by $(PBS)_n$, if the B-tensor of type $(0, 2)$ is nonzero and satisfies the condition

$$\nabla_k B_{ij} = 2A_k B_{ij} + A_i B_{kj} + A_j B_{ik}, \tag{1.6}$$

where A_i is a nonzero 1-form. Obviously, one can see that for $a = 1$ and $b = \frac{\phi}{R}$, the $(PBS)_n$ reduces to $(PZS)_n$ ([19, 22]) and for $a = 1$ and $b = 0$, the $(PBS)_n$ reduces to pseudo-Ricci symmetric manifolds [3].

On the other hand, quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. A non-flat pseudo-Riemannian manifold $(M^n, g)(n > 2)$ is defined to be a quasi Einstein manifold [4] if its Ricci tensor R_{ij} of type $(0, 2)$ is not identically zero and satisfies the following condition:

$$R_{ij} = \alpha g_{ij} + \beta \eta_i \eta_j,$$

where α, β are scalars and η_i is a nonzero 1-form for all vector fields X. The quasi-Einstein manifold is denoted by $(QE)_n$.

Gray [11] introduced the notion of cyclic parallel Ricci tensor and Codazzi-type of Ricci tensor. A pseudo-Riemannian manifold is said to satisfy cyclic parallel Ricci tensor [11] if its Ricci tensor R_{ij} of type $(0, 2)$ is nonzero and satisfies the condition

$$\nabla_k R_{ij} + \nabla_i R_{kj} + \nabla_j R_{ik} = 0. \tag{1.7}$$

Again, a pseudo-Riemannian manifold is said to satisfy Codazzi-type of Ricci tensor if its Ricci tensor R_{ij} of type $(0, 2)$ is nonzero and satisfy the following condition:

$$\nabla_k R_{ij} = \nabla_j R_{ik}. \tag{1.8}$$

We also have a very useful lemma as follows.

Walker's Lemma ([28]). *If a_{ij}, b_{ij} are numbers satisfying $a_{ij} = a_{ji}$, and $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$ for $i, j, k = 1, 2, \ldots, n$, then either all $a_{ij} = 0$ or, all b_i are zero.*

The paper is organized as follows: After preliminaries in Sec. 3, we prove the existence of a $(PBS)_n(n > 2)$. In Sec. 4, we consider B-semisymmetric manifolds. Next, we obtain a sufficient condition for a $(PBS)_n$ to be pseudo-Ricci symmetric in the sense of Deszcz [10]. In Sec. 6, we consider a $(PBS)_n(n > 2)$ with cyclic parallel B-tensor and Codazzi-type of B-tensor. Finally, we consider conformally flat $(PBS)_n$.

Throughout the paper, all manifolds under consideration are assumed to be connected Hausdorff manifolds endowed with a non-degenerate metric of arbitrary signature, that is, n-dimensional pseudo-Riemannian manifolds. Particularly, we will take into consideration n-dimensional Lorentzian manifolds, that is, with metrics of signature $s = n - 2$ [13].

2. Preliminaries

In this section, we study some well-known structures on pseudo-Riemannian manifolds satisfied by B-tensor as follows:

(i) If $B_{ij} = 0$ (the B-flat manifold), then the manifold is an Einstein manifold [1], $R_{ij} = -\frac{bR}{a} g_{ij}$.

(ii) If $\nabla_k B_{ij} = \lambda_k B_{ij}$, (the B-recurrent manifold) then the manifold is a generalized Ricci-recurrent manifold [8]. The condition is equivalent to

$$\nabla_k R_{ij} = \mu_k R_{ij} + (n-1)\gamma_k g_{ij},$$

where $\mu_k = -\frac{\nabla_k a}{a} + a\lambda_k$ and $\gamma_k = -(R\nabla_k b + \nabla_k R b) + \lambda_k bR$.

If $\mu_k = 1$ and $\gamma_k = 0$, then the manifold reduces to a Ricci recurrent manifold.

(iii) Einstein equation [7] with cosmological constant λ and energy–stress tensor T_{kl} may be written as

$$\frac{1}{a} B_{ij} = \kappa T_{ij},$$

where $\frac{bR}{a} = -\frac{1}{2}R + \lambda, a \neq 0$ and κ is the gravitational constant. Then, $\frac{1}{a}$ times of B_{ij} tensor may be thought of as a generalized Einstein gravitational tensor with arbitrary scalar function $\frac{bR}{a}$.

Various conditions on the energy–momentum tensor determine constraints on the B-tensor. The vacuum solution $B = 0$ determines an Einstein space with $\lambda = \frac{n-2}{2n} R$; conservation of total energy–momentum ($\nabla_l T_{kl} = 0$) implies that

$$\nabla_l \left(\frac{1}{a}\right) B_{kl} + \frac{1}{a} \nabla^l B_{kl} = 0$$

and

$$\nabla_k \left\{\left(\frac{1}{2} + \frac{b}{a}\right)\right\} = 0;$$

the condition $\nabla_i B_{kl} = 0$ describes a space-time with conserved energy–momentum density.

3. Existence of a $(PBS)_n (n > 2)$

In this section, it is shown that there exists a pseudo-Riemannian manifold $(M^n, g)(n \geq 2)$, where B tensor satisfies the condition (1.1) and for which $\nabla_i B_{jk} \neq 0$.

For this, we consider a pseudo-Riemannian manifold $(M^n, g)(n \geq 2)$ which admits a linear connection $\bar{\Gamma}^h_{ij}$ defined by

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + A_i \delta^h_j + A_j \delta^h_i, \tag{3.1}$$

where A_i is a nonzero 1-form and which is such that

$$\bar{\nabla}_i B_{jk} = 0, \tag{3.2}$$

where $\bar{\nabla}_i$ denotes the covariant differentiation with respect to the connection $\bar{\Gamma}^h_{ij}$. If (3.2) is to hold, then we obtain

$$\frac{\partial}{\partial x^i} B_{jk} - B_{hk} \bar{\Gamma}^h_{ji} - B_{jh} \bar{\Gamma}^h_{ki} = 0. \tag{3.3}$$

Using (3.1) in (3.3), we get

$$\frac{\partial}{\partial x^i} B_{jk} - B_{hk}(\Gamma^h_{ji} + A_j \delta^h_i + A_i \delta^h_j) - B_{jh}(\Gamma^h_{ki} + A_k \delta^h_i + A_i \delta^h_k) = 0. \tag{3.4}$$

From (3.4), we obtain

$$\nabla_i B_{jk} = 2A_i B_{jk} + A_j B_{ik} + A_k B_{ji}. \tag{3.5}$$

The connection $\bar{\nabla}$ is not identical with ∇. Hence, $\nabla_i B_{jk} \neq 0$. Thus, if a pseudo-Riemannian manifold $(M^n, g)(n \geq 2)$ admits a linear connection $\bar{\nabla}$ which satisfies (3.1) and (3.2), then the manifold is a $(PBS)_n$.

Hence, we have the following.

Theorem 3.1. *If a pseudo-Riemannian manifold* $(M^n, g)(n \geq 2)$ *admits a linear connection* $\bar{\nabla}$ *which satisfies* (3.1) *and* (3.2), *then the manifold is a* $(PBS)_n(n \geq 2)$.

4. *B*-Semisymmetric Manifolds

A pseudo-Riemannian manifold is said to be Ricci-semisymmetric if $R \circ S = 0$ holds, that is, $(R(X, Y) \circ S)(U, V) = 0$ for all vector fields X, Y, U and V, where $R(X, Y)$ denotes the curvature operator and S is the Ricci tensor of type $(0, 2)$, which can be rewritten in local coordinate system as $(R \circ S)_{ijlm} = 0$, where $(R \circ S)_{ijlm} = R_{rj} R^r_{ilm} + R_{ri} R^r_{jlm}$ and R_{ij} and R^l_{ijk} are local components of Ricci tensor S of type $(0, 2)$ and Riemann curvature tensor R of type $(1, 3)$, respectively. Analogous to this definition, we define B-semisymmetric manifold. A pseudo-Riemannian manifold is said to be B-semisymmetric if $(R \circ B)_{ijlm} = 0$.

In this section, we consider a B-semisymmetric manifold. Thus, we have

$$(R \circ B)_{ijlm} = 0. \tag{4.1}$$

Now,

$$(R \circ B)_{ijlm} = B_{rj} R^r_{ilm} + B_{ri} R^r_{jlm}. \tag{4.2}$$

Using (4.1) in (4.2), we get

$$B_{rj}R^r_{ilm} + B_{ri}R^r_{jlm} = 0. \tag{4.3}$$

From (4.3) and (1.4), we obtain

$$a(R_{rj}R^r_{ilm} + R_{ri}R^r_{jlm}) + bR(g_{rj}R^r_{ilm} + g_{ri}R^r_{jlm}) = 0, \tag{4.4}$$

which implies

$$a(R_{rj}R^r_{ilm} + R_{ri}R^r_{jlm}) = 0. \tag{4.5}$$

Since $a \neq 0$, thus (4.5) can be rewritten as follows:

$$R_{rj}R^r_{ilm} + R_{ri}R^r_{jlm} = 0, \tag{4.6}$$

which implies

$$(R \circ S)_{ijlm} = 0. \tag{4.7}$$

Hence, the manifold is a Ricci-semisymmetric manifold. Conversely, if (4.7) holds, then from (4.2), we can conclude that (4.1) holds, that is, Ricci-semisymmetry implies B-semisymmetry.

Thus, we have the following.

Theorem 4.1. *A pseudo-Riemannian manifold is B-semisymmetric if and only if it is Ricci-semisymmetric.*

5. Sufficient Conditions for a $(PBS)_n (n > 2)$ to be Ricci Pseudo-Symmetric in the Sense of Deszcz

In this section, we investigate sufficient conditions for pseudo-B-symmetric manifolds to be Ricci pseudo-symmetric in the sense of Deszcz.

We have from (1.6)

$$\nabla_s B_{kl} = 2A_s B_{kl} + A_k B_{sl} + A_l B_{sk}. \tag{5.1}$$

Taking covariant derivative on (5.1), we get

$$\begin{aligned}\nabla_i\nabla_s B_{kl} = {} & 2(\nabla_i A_s)B_{kl} + 2A_s(2A_i B_{kl} + A_k B_{il} + A_l B_{ik}) + (\nabla_i A_k)B_{sl} \\ & + A_k(2A_i B_{sl} + A_s B_{il} + A_l B_{is}) + (\nabla_i A_l)B_{sk} + A_l(2A_i B_{sk} \\ & + A_s B_{ik} + A_k B_{is}).\end{aligned} \tag{5.2}$$

Interchanging the indices s and i in (5.2) and subtracting, we obtain

$$\begin{aligned}(\nabla_s\nabla_i - \nabla_i\nabla_s)B_{kl} = {} & 2(\nabla_s A_i - \nabla_i A_s)B_{kl} + B_{il}(\nabla_s A_k - A_k A_s) \\ & - B_{sl}(\nabla_i A_k - A_k A_i) + B_{ki}(\nabla_s A_l - A_l A_s) \\ & - B_{sk}(\nabla_i A_l - A_l A_i).\end{aligned} \tag{5.3}$$

Now, if possible let

$$\nabla_s A_k = A_k A_s + \gamma g_{ks}, \tag{5.4}$$

where γ is an arbitrary scalar function. Then, we have

$$(\nabla_s \nabla_i - \nabla_i \nabla_s) B_{kl} = \gamma (B_{il} g_{sk} - B_{sl} g_{ik} + B_{ki} g_{sl} - B_{sk} g_{il}). \tag{5.5}$$

Now, using (1.4) in (5.5) yields

$$(\nabla_s \nabla_i - \nabla_i \nabla_s) B_{kl} = \gamma (R_{il} g_{sk} - R_{sl} g_{ik} + R_{ki} g_{sl} - R_{sk} g_{il}). \tag{5.6}$$

If (5.6) holds, then we call the manifold pseudo-Ricci symmetric in the sense of Deszcz [10]. Thus, we have the following.

Theorem 5.1. *If M is an n-dimensional $(PBS)_n$ and the associated 1-form is concircular of the form $\nabla_s A_k = A_k A_s + \gamma g_{ks}$, then the manifold is pseudo-Ricci symmetric in the sense of Deszcz.*

On the other hand, if we consider a pseudo-B-symmetric manifold, which is also pseudo-Ricci symmetric in the sense of Deszcz [10], then we can obtain an interesting result.

From the contracted second Bianchi identity $\nabla_m R^m_{jkl} = \nabla_k R_{jl} - \nabla_j R_{kl}$ and from the definition of the B-tensor, we have

$$a \nabla_m R^m_{jkl} = \nabla_k B_{jl} - \nabla_j B_{kl} + [(\nabla_j (bR)) g_{kl} - (\nabla_k (bR)) g_{jl}]. \tag{5.7}$$

From (1.6) and (5.7), we get

$$a \nabla_m R^m_{jkl} = A_k B_{jl} - A_j B_{kl} + [(\nabla_j (bR)) g_{kl} - (\nabla_k (bR)) g_{jl}]. \tag{5.8}$$

Taking covariant derivative of (5.8) yields

$$\begin{aligned} \nabla_i a \nabla_m R^m_{jkl} + a \nabla_i \nabla_m R^m_{jkl} &= (\nabla_i A_k) B_{jl} + A_k (\nabla_i B_{jl}) - (\nabla_i A_j) B_{kl} - A_j (\nabla_i B_{kl}) \\ &\quad + [(\nabla_i \nabla_j (bR)) g_{kl} - (\nabla_i \nabla_k (bR)) g_{jl}]. \end{aligned} \tag{5.9}$$

By performing a cyclic permutation of indices i, j, k and then adding the resulting three equations and using the contracted Bianchi identity, we obtain

$$\begin{aligned} &\nabla_i a \nabla_m R^m_{jkl} + \nabla_j a \nabla_m R^m_{kil} + \nabla_k a \nabla_m R^m_{ijl} \\ &\quad + a[(\nabla_i \nabla_k - \nabla_k \nabla_i) R_{jl} + (\nabla_j \nabla_i - \nabla_i \nabla_j) R_{kl} + (\nabla_k \nabla_j - \nabla_j \nabla_k) R_{il}] \\ &= (\nabla_i A_k - \nabla_k A_i) B_{jl} + (\nabla_j A_i - \nabla_i A_j) B_{kl} + (\nabla_k A_j - \nabla_j A_k) B_{il}]. \end{aligned} \tag{5.10}$$

Now if the manifold is pseudo-Ricci symmetric in the sense of Deszcz [10], then from (5.10), we obtain

$$\begin{aligned} &(\nabla_i A_k - \nabla_k A_i) B_{jl} + (\nabla_j A_i - \nabla_i A_j) B_{kl} + (\nabla_k A_j - \nabla_j A_k) B_{il} \\ &\quad = \nabla_i a \nabla_m R^m_{jkl} + \nabla_j a \nabla_m R^m_{kil} + \nabla_k a \nabla_m R^m_{ijl}. \end{aligned} \tag{5.11}$$

Suppose a is constant, then (5.11) reduces to

$$(\nabla_i A_k - \nabla_k A_i) B_{jl} + (\nabla_j A_i - \nabla_i A_j) B_{kl} + (\nabla_k A_j - \nabla_j A_k) B_{il} = 0. \tag{5.12}$$

Now if $\det(B_{kl}) \neq 0$, then there exists a $(2,0)$ tensor $(B^{-1})^{km}$ with the property $B_{kl}(B^{-1})^{km} = \delta_l^m$.

Multiplying (5.12) by $(B^{-1})^{hl}$, we obtain

$$(\nabla_i A_k - \nabla_k A_i)\delta_j^h + (\nabla_j A_i - \nabla_i A_j)\delta_k^h + (\nabla_k A_j - \nabla_j A_k)\delta_i^h = 0. \tag{5.13}$$

Putting $h = j$ and summing from (5.13) yields

$$(n-2)(\nabla_i A_k - \nabla_k A_i) = 0. \tag{5.14}$$

Thus for $n > 2$, the 1-form A_k is a closed 1-form. Hence, we have the following.

Theorem 5.2. *If a $(PBS)_n(n > 2)$ is pseudo-Ricci symmetric in the sense of Deszcz and a is constant, then the associated 1-form A is closed, provided the B-tensor is non-singular.*

6. $(PBS)_n(n > 2)$ with Cyclic Parallel B-Tensor and Codazzi Type of B-Tensor

In analogy to the definition in (1.7), we define cyclic B-tensor as follows.

An n-dimensional manifold is said to be cyclic B-tensor if the following condition holds:

$$\nabla_k B_{ij} + \nabla_i B_{kj} + \nabla_j B_{ik} = 0. \tag{6.1}$$

Now from (1.6), we obtain

$$\nabla_k B_{ij} + \nabla_i B_{kj} + \nabla_j B_{ik} = 4A_k B_{ij} + 4A_i B_{kj} + 4A_j B_{ik}. \tag{6.2}$$

Using (6.1) in (6.2) yields

$$4A_k B_{ij} + 4A_i B_{kj} + 4A_j B_{ik} = 0. \tag{6.3}$$

Then by Walker's lemma, we can see that either $A_i = 0$ or $B_{ij} = 0$ for all i, j. But both of A_i and B_{ij} are not zero in a $(PBS)_n$. Hence, we have the following.

Theorem 6.1. *There does not exist a $(PBS)_n(n > 2)$ with cyclic parallel B-tensor.*

Now, we suppose that the B-tensor in a $(PBS)_n(n > 2)$ is of Codazzi type. Now from (1.6), we obtain

$$\nabla_k B_{jl} - \nabla_j B_{kl} = A_k B_{jl} - A_j B_{kl}. \tag{6.4}$$

Since B is of Codazzi type, we have from (6.4)

$$A_k B_{jl} - A_j B_{kl} = 0. \tag{6.5}$$

Now multiplying (6.5) by A^k and taking sum, we get

$$A^k A_k B_{jl} - A_j A^k B_{kl} = 0. \tag{6.6}$$

Again transvecting (6.5) by g^{jl} yields

$$A_k B - A^l B_{kl} = 0. \tag{6.7}$$

Using (6.7) in (6.6), we get

$$B_{jl} = \frac{A_j A_l}{A^k A_k} B. \tag{6.8}$$

Using (1.4), (1.5) in (6.8) and simplifying, we obtain

$$R_{jl} = -\frac{bR}{a} g_{jl} + \frac{(a+bn)}{a} R E_j E_l, \tag{6.9}$$

where $E_j = \frac{A_j}{\|A\|}$.

We rewrite (6.9) as follows:

$$R_{jl} = \alpha g_{jl} + \beta E_j E_l, \tag{6.10}$$

where $\alpha = -\frac{bR}{a}$ and $\beta = \frac{(a+bn)}{a} R$. Thus, we have the following.

Theorem 6.2. *A $(PBS)_n (n > 2)$ with Codazzi type of B-tensor is a quasi-Einstein manifold.*

Remark 1. The above theorem generalizes the results of [9].

Moreover from (6.5) and definition (1.6), we have $\nabla_k B_{ij} = 2A_k B_{ij} + 2A_i B_{kj} = 4A_k B_{ij}$ and the tensor B_{ij} is recurrent.

Theorem 6.3. *Let M be an $n(n > 3)$ dimensional $(PBS)_n$ pseudo-Riemannian manifold: if the condition $\nabla_k B_{jl} = \nabla_j B_{kl}$ is satisfied, then the tensor B_{ij} is recurrent, that is, $\nabla_k B_{jl} = 4A_k B_{jl}$.*

The case in which the vector A_k results to be a null vector, that is, $A^j A_j = 0$ is even more interesting. Let θ^k be a vector such that $\theta^k A_k = 1$: from $A_j B_{kl} = A_k B_{jl}$ we have $B_{kl} = A_k \theta^j B_{jl}$ and by symmetry also $A_k \theta^j B_{jl} = A_l \theta^j B_{jk}$ and thus $\theta^j B_{jl} = A_l (\theta^k \theta^j B_{kj})$ from which finally:

$$B_{kl} = \psi A_k A_l, \tag{6.11}$$

being $\theta^m \theta^j B_{mj} = \psi$ a scalar function. The rank of the tensor B_{kl} is thus one. Contracting (6.11) with g^{kl}, we get $B = 0$, so that $R = 0$ or $b = -\frac{a}{n}$. In the first case, the Ricci tensor is given by $R_{kl} = \frac{\psi}{a} A_k A_l$ and its rank is one; in the second case, the Ricci tensor turns out to be $R_{kl} = \frac{\psi}{a} A_k A_l + \frac{R}{n} g_{kl}$. The following theorem may be stated.

Theorem 6.4. *Let M be an $n(n > 3)$-dimensional $(PBS)_n$ pseudo-Riemannian manifold: if the condition $\nabla_k B_{jl} = \nabla_j B_{kl}$ is satisfied, and the vector A_j results to be a null vector, that is, $A_j A^j = 0$, then the Ricci tensor takes the form $R_{kl} = \frac{\psi}{a} A_k A_l$ or $R_{kl} = \frac{\psi}{a} A_k A_l + \frac{R}{n} g_{kl}$.*

We follow now a trick due to Roter in [24], Theorem 1. Inserting (6.11) in $\nabla_k B_{jl} = 4A_k B_{jl}$ after a straightforward calculation, we infer:

$$(\nabla_j A_k) A_l + A_k (\nabla_j A_l) = [4A_j - \nabla_j \ln|\psi|] A_k A_l. \tag{6.12}$$

On multiplying the previous result by θ^k, we get easily:

$$(\nabla_j A_k) + A_k \theta^l (\nabla_j A_l) = [4A_j - \nabla_j \ln|\psi|] A_k. \tag{6.13}$$

Again a multiplication by θ^k gives:

$$(\nabla_j A_k)\theta^k = \frac{1}{2}[4A_j - \nabla_j \ln|\psi|] A_k, \tag{6.14}$$

and inserting back in (6.13) the covector A_j results to be recurrent, that is,

$$\nabla_j A_k = \frac{1}{2}[4A_j - \nabla_j \ln|\psi|] A_k = p_j A_k. \tag{6.15}$$

If the covector A_j is closed, then from the recurrence relation we get $p_j A_k = p_k A_j$ and transvecting this with θ^k it is easily seen that $p_j = \gamma A_j$ for some function γ and thus,

$$\nabla_j A_k = \gamma A_j A_k. \tag{6.16}$$

Now let us suppose that the one form A_k is locally a gradient, that is, $A_j = \nabla_j h$ for some scalar function h on the manifold: it can be see easily that the rescaled null covector $\bar{A}_k = A_k e^{-\frac{1}{2}[4h - \ln|\psi|]}$ is a covariantly constant, that is, $\nabla_j \bar{A}_k = 0$; we have proved the following.

Theorem 6.5. *Let M be an $n(n > 3)$-dimensional $(PBS)_n$ pseudo-Riemannian manifold: if the condition $\nabla_k B_{jl} = \nabla_j B_{kl}$ is satisfied and the vector A_j satisfies $A^j A_j = 0$ then the null covector A_k is recurrent, that is, $\nabla_j A_k = p_j A_k$ for some one form p_j; further if the same covector is locally a gradient, then it can be rescaled to a null covariant constant.*

Lorentzian manifolds, that is, space-times with recurrent null vectors were studied for a long time (see for example [2, 14, 15, 25, 27]). In particular, Walker [27] found a set of canonical coordinates for the metric in such case. Here, we refer to [15, Proposition 1].

Theorem 6.6. *Let (M, g) be a Lorentzian manifold of dimension $n + 2 > 2$ with a recurrent null vector field $\nabla_k X_j = pX_j$.*

(1) *This is equivalent to the existence of coordinates $(v, x_1, \ldots, x_n, u)$ in which the metric has the following local shape:*

$$\begin{aligned} ds^2 = {} & 2du dv + a_i(x_1, \ldots, x_n, u) dx^i du + H(v, x_1, \ldots, x_n, u) du^2 \\ & + g(x_1, \ldots, x_n, u) dx^i dx^j \end{aligned} \tag{6.17}$$

with $\frac{\partial g_{ij}}{\partial v} = \frac{\partial a_i}{\partial v} = 0, H \in C^\infty(M)$. To these coordinates, we refer as Walker coordinates.

(2) *$\nabla_k X_j = 0$ if and only if H does not depend on v, that is, $\frac{\partial H}{\partial v} = 0$. To these coordinates, we refer as Brinkmann coordinates.*

A Lorentzian manifold with null covariantly constant vector field is named Brinkmann wave after [2]. In [17], an n-dimensional pseudo-Riemannian manifold on which the Ricci tensor has the form $R_{kl} = \psi X_k X_l$ and the null vector X_k is recurrent, that is, $\nabla_k X_j = p_k X_j$, is named pure radiation metric with parallel rays or aligned pure radiation metric. In view of Theorem 6.3, we can thus state the following.

Theorem 6.7. *Let M be an $n(n > 3)$-dimensional $(PBS)_n$ space-time: if the condition $\nabla_k B_{jl} = \nabla_j B_{kl}$ is satisfied and the vector satisfies $A^j A_j = 0$, then the metric assumes the local shape* (6.17) *in Walker coordinates*; *further if the null vector A_k is locally a gradient*, *then the manifold is a Brinkmann wave.*

These results generalize similar ones in [23].

7. Conformally Flat $(PBS)_n(n > 2)$

In general, the B-tensor in a $(PBS)_n$ is not of Codazzi type. In this section, it is shown that the B-tensor in a conformally flat $(PBS)_n(n > 2)$ is of Codazzi type. It is known that in a conformally flat manifold, the following relation holds:

$$\nabla_k R_{ij} - \nabla_j R_{ik} = \frac{1}{2(n-1)}[g_{ij}\nabla_k R - g_{ik}\nabla_j R]. \tag{7.1}$$

Here, we consider a conformally flat $(PBS)_n(n > 2)$. Now from (1.4), we obtain

$$\nabla_k R_{ij} - \nabla_j R_{ik} = \nabla_k B_{ij} - \nabla_j B_{ik}. \tag{7.2}$$

Let a, b and R be constants.

Then from (7.1) and (7.2), we get

$$\nabla_k B_{ij} - \nabla_j B_{ik} = 0.$$

Thus B-tensor in the $(PBS)_n(n > 2)$ of Codazzi type. Hence, we have the following.

Theorem 7.1. *The B-tensor in a conformally flat $(PBS)_n(n > 2)$ with constant value of a, b and R is of Codazzi type.*

Next, we consider conformally flat $(PBS)_n$ Lorentzian manifolds with a,b, $b \neq -\frac{a}{n}$ constants and vanishing of the scalar curvature, that is, $R = 0$. Then, $\nabla_k B_{jl} = \nabla_j B_{kl}$ and if the vector A_j satisfies $A^j A_j = 0$, the Ricci tensor writes as $R_{kl} = \frac{\psi}{a} A_k A_l$. Moreover, from Theorem 6.5, the null vector is recurrent, that is, $\nabla_j A_k = p_j A_k$.

Now, we introduce the definition of a *pp*-wave and related properties as stated in ([14–16]).

Definition 7.1 ([14–16]). *A Brinkmann wave is called pp-wave if its curvature tensor satisfies the trace condition $R_{jk}^{pq} R_{pqlm} = 0$.*

In [25], the following coordinate description and equivalence are proved. Here, we remand to ([14–16]).

Lemma 7.1 ([14–16, 25]). *A Lorentzian manifold* (M, g) *of dimension* $n+2>2$ *is a pp-wave if and only if there exist coordinates* $(v, x_1, \ldots, x_n, u)$ *in which the metric has the following local shape*:

$$ds^2 = 2dudv + H(x_1, \ldots, x_n, u)du^2 + dx_j dx^j, \tag{7.3}$$

where $H(x_1, \ldots, x_n, u)$ *is an arbitrary smooth function with the property* $\frac{\partial H}{\partial v} = 0$, *usually called the potential function of the pp-wave.*

Lemma 7.2 ([14–16, 25]). *A Lorentzian manifold* (M, g) *of dimension* $n+2>2$ *with parallel null vector field* $\nabla_k X_= 0$ *is a pp-wave if and only if one of the following conditions is satisfied*:

$$X_i R_{jklm} + X_j R_{kilm} + X_k R_{ijlm} = 0, \tag{7.4}$$

$$R_{jklm} = X_j X_m D_{kl} - X_j X_l D_{mk} - X_k X_m D_{jl} + X_k X_l D_{jm}, \tag{7.5}$$

$$R^{pq}_{jk} R_{plmq} = \chi X_j X_k X_l X_m, \tag{7.6}$$

being D_{ij} *a symmetric tensor and* χ *a suitable scalar function. The Ricci tensor of a pp-wave is given by* $R_{kl} = \psi X_k X_l$ *for a smooth function* ψ. *In dimension* $n = 4$, *this is even equivalent to* $R^{pq}_{jk} R_{plmq} = 0$ (*see* [18]).

As a first from the definition of the conformal curvature tensor and from the local form of the Ricci tensor, the following relation is displayed immediately:

$$A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = A_i R_{jklm} + A_j R_{kilm} + A_k R_{ijlm}. \tag{7.7}$$

Transvecting the previous equation by g^{im} and taking account of $R_{kl} = \frac{\psi}{a} A_k A_l$, we easily get $A^m C_{jklm} = A^m R_{jklm}$. Since the space is conformally flat, we have $A_i R_{jklm} + A_j R_{kilm} + A_k R_{ijlm} = 0$ from (7.7) and $A^m R_{jklm} = 0$. A skew symmetrization of the covariant derivative of the recurrence condition $\nabla_j A_k = p_j A_k$ and the Ricci identity give $R^m_{jkl} A_m = (\nabla_j A_k - \nabla_k A_j) A_l$. This result ensures that, at least locally, A_k(see [12, pp. 242–243]) is a gradient, that is, $A_j = \nabla_j h$ and thus such covector can be locally rescaled to a null covariantly constant $\bar{A}_k = A_k e^{-\frac{1}{2}[4h - ln|\psi|]}$ so that $\nabla_j \bar{A}_k = 0$ and $\bar{A}_i R_{jklm} + \bar{A}_j R_{kilm} + \bar{A}_k R_{ijlm} = 0$. Lemma 7.2. ensures that the metric is (7.3) and thus *pp*-wave metric.

Theorem 7.2. *Let* M *be a conformally flat* n*-dimensional* $(PBS)_n$ *space-time with* $a, b,$ $b \neq -\frac{a}{n}$ *constants and vanishing of the scalar curvature, that is,* $R = 0$: *if* $A_k A^k = 0$ *then* A_j *is locally a gradient and can be rescaled to a covariantly constant vector* $\bar{A}_j$, *the relation* $\bar{A}_i R_{jklm} + \bar{A}_j R_{kilm} + \bar{A}_k R_{ijlm} = 0$ *holds and the space is thus a pp-wave with metric* (7.3).

These results generalize similar ones in [23].

Acknowledgment

The first author was supported by Grant Project No. NRF-2015-R1A2A1A-01002459 from National Research Foundation.

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